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Nice Polynomials

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NICE POLYNOMIALS

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NICE POLYNOMIALS

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Abstract

We consider the problem of finding, constructing, and classifying nice polynomials. After a short history of previous results, we present a general property of nice polynomials which leads to an important modification of the concept of equivalence classes of nice polynomials. We give several important results on nice symmetric or anti-symmetric polynomials with an odd number of roots, which dramatically increase the speed of a computer search for examples. We present complete solutions to the symmetric three root case, the general three root case, and the symmetric four root case. We also give the relations between the roots and critical points for the general four root case and the symmetric five, six, and seven root cases. Using the relations for the general three and four root cases, we state, without proof, the suggested pattern for the relations for the general N root case. We present several important examples found by a computer search, including the first known examples of nice symmetric or antisymmetric polynomials with five distinct roots and the first known examples of nice polynomials with six distinct roots. To conclude our study, we present several open problems and new conjectures suggested by our results, examples, and computer searches.

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. Some Previous Results

Nice polynomials are polynomials whose coefficients, roots, and critical points are integers. M. Chapple published the first known paper [**4**] on nice polynomials in 1960. Several minor publications followed, including several independent papers on the cubic case [**1**] and [**7**]. In 1999, Richard Nowakowski added the problem of finding, constructing, and classifying nice polynomials to the list of unsolved problems [**6**] in *The American Mathematicial Monthly.* In 2000, Buchholz and MacDougall published the main paper [**2**] on this subject. We will not discuss many results from this paper since the authors often consider derivatives of higher order than the first. In fact, they often work with *totally nice* polynomials; but, for simplicity, we do not consider such polynomials. (A polynomial $p(x)$ of degree d is *totally nice* if the roots of $p(x)$, $p'(x), \ldots, p^{(d-1)}(x)$ are integers.) In a submitted paper, Evard [**5**] takes a completely different approach by considering the relations between the roots and critical points of polynomials (see Lemma 1.3.1), and he generalizes some of these results from $\mathbb{Z}$ to integral domains. For simplicity, we require $p(x) \in \mathbb{Z}[x]$.

Only [**2**, Theorem 1], [**5**, Formulas A10-A11], and [**5**, Examples B7-B8] contain results about nice quintics, and no results about nice polynomials of higher degree (except [**2**, Theorem 2]) have been published so far.

We will discuss these previous results in more detail at the beginnings of the appropriate chapters.

1.2. Equivalence Transformations

While we often work with nice polynomials $p(x)$, in some cases we may require $p(x)$ to have rational coefficients, roots, and critical points. Such polynomials we call *Q-nice.* The existence of nice and Q-nice polynomials is equivalent (which we will see below). Whether we consider nice or Q-nice polynomials depends on the context in which we are working. For example, if we work with nice polynomials, then we considerably reduce the amount of work of a computer search for examples (unless we have a formula that gives all examples we are looking for). Furthermore, we can discuss the prime factorization of the roots and critical points. However, for certain cases, working with Q-nice polynomials considerably simplifies the work of finding formulas. See Section 3 in Chapter 3 and Section 3 in Chapter 4 for examples of these cases.

Note that a nice (or Q-nice) polynomial of any degree d exists since $p(x) = (x-r)^d$ is nice (or Q-nice) for any integer (or rational) r. We call such nice (or Q-nice) polynomials *trivial.*

The *type* of a polynomial is specified by the multiplicities of the roots without regard to the size of the roots. For example, $p_1(x) = x^3(x-1)^2(x+5)^4$ and $p_2(x) = (x+1)^4(x-7)^2(x+6)^3$ are two examples of the same type, but $p_3(x) = x^2(x-1)^3(x+2)$ and $p_4(x) = x(x+2)^3(x-1)$ are not.

To count the number of nice (or Q-nice) polynomials of a given type, we first note that certain simple transformations [**5**, Proposition 2.1], including one noted by the author, transform nice polynomials into other nice polynomials, which we will call *equivalent*. Shifting a nice polynomial $p(x)$ to the left or right $k \in \mathbb{N}$ units results in an equivalent nice polynomial, and stretching or shrinking $p(x)$ by a factor of $k \in \mathbb{N}$ also results in an equivalent nice polynomial. Reflecting $p(x)$ over one of the coordinate axes also results in an equivalent nice polynomial. If $p(x)$ is Q-nice, then shifting $p(x)$ to the left or right $k \in \mathbb{Q}$ units, stretching or shrinking $p(x)$ by a factor of $k \in \mathbb{Q}$, or reflecting $p(x)$ over one of the coordinate axes result in an equivalent Q-nice polynomial. Because the stretch or shrink is an equivalence transformation, we now see that the existence of nice and Q-nice polynomials is equivalent; that is, for every Q-nice polynomial, there exists an equivalent nice polynomial and vice-versa. Furthermore, raising a nice polynomial $p(x)$ to the nth power also results in an equivalent nice (or Q-nice) polynomial. This transformation is formally stated in the theorem below.

THEOREM 1.2.1. *Let $p(x)$ be a polynomial with integer coefficients. Then, for all natural numbers n, $p(x)$ is nice (or Q-nice) iff $[p(x)]^n$ is nice (or Q-nice).*

PROOF. Assume $p(x)$ is nice. Since $p(x)$ and $[p(x)]^n$ have the same roots, $[p(x)]^n$ also has integer roots. If we differentiate $[p(x)]^n$, we have $\frac{d}{dx}[p(x)]^n = n[p(x)]^{n-1}p'(x)$. Since the roots of $[p(x)]^{n-1}$ and $p'(x)$ are integers, we conclude that $[p(x)]^n$ is also nice. By the same reasoning, if $p(x)$ is Q-nice, then $[p(x)]^n$ is also Q-nice.

For the reverse direction, assume $q(x) = [p(x)]^n$ is nice. Raising a polynomial $p(x)$ to the nth power does not change the union of the sets of roots and critical points. Therefore, if all the roots and critical points of $q(x)$ are integers, then all the roots and critical points of $p(x)$ are also integers. Likewise, if $q(x)$ is Q-nice, then $p(x)$ is Q-nice. ■

Remark. The power transformation as an equivalence transformation was recently noted by the author. Though $p(x)$ and $[p(x)]^n$ are not equivalent geometrically (since they do not have similar shapes), they are equivalent algebraically since the equations for $p(x)$ and $[p(x)]^n$ do not differ significantly; for this reason, we consider $p(x)$ and $[p(x)]^n$ equivalent nice (or Q-nice) polynomials.

In general, we say that nice (or Q-nice) polynomials $p_1(x)$ and $p_2(x)$ are *equivalent* whenever $p_1(x)$ can be transformed into $p_2(x)$ by a finite composition of the following transformations: horizontal shifts, reflections over any axis, stretching or shrinking (see [**5**, Proposition 2.1]), or the power transformation mentioned above. Therefore, whenever we count the number of nice polynomials of a certain type, we count the number of equivalence classes, not the actual number of examples of nice polynomials since there are infinitely many such polynomials in every equivalence class.

Assumptions. Because of the stretch or shrink, we may assume that all the greatest common divisor of all the nonzero roots and critical points of $p(x)$ equals 1. In this case, we say that $p(x)$ is in *reduced form*. We may also assume all nice (or Q-nice) polynomials are monic. Furthermore, because the power transformation is an equivalence transformation, we may take the nth root of $p(x)$ if the greatest common divisor of all the multiplicities of the roots equals n. Therefore, we may assume, unless otherwise stated, that the greatest common divisor of all the multiplicities of the roots of $p(x)$ equals 1.

1.3. Relations Between the Roots and Critical Points

The key tool to use when working with nice polynomials is the relations between the roots and critical points of polynomials, which is stated in the lemma below. This approach is the one that Evard uses in his paper [**5**].

LEMMA 1.3.1. *Let $p(x)$ be a polynomial of degree d with rational coefficients and with d rational roots $r_1, r_2, ..., r_d$. Then $p(x)$ is Q-nice iff there exist rational numbers $c_1, c_2, ..., c_{d-1}$ such that*

$$(d-k)S_k(r_1, r_2, ..., r_d) = dS_k(c_1, c_2, ..., c_{d-1}), \tag{1.3.1}$$

for all $k \in \mathbb{N}$ such that $1 \leq k \leq d-1$ and S_k, the kth elementary symmetric polynomial, is defined as follows:

$$S_k(r_1, ..., r_d) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq d} r_{i_1} r_{i_2} ... r_{i_k}. \tag{1.3.2}$$

PROOF. This result is a direct consequence of [**5**, Corollary 3.2]. ■

In the next chapter, we present several important results on nice symmetric or antisymmetric polynomials with an odd number of distinct roots. Although these results are not complete, they considerably reduce the amount of work of a computer search for examples. In Chapter 3, we present a complete solution to the symmetric three root case. As a contrast to the symmetric three root case, we present the complete solution to the general three root case—noting the differences between the symmetric case and the general case. We also give examples to illustrate our results. In Chapter 4, we give the relations between the roots and critical points for the general four root case, then we present the complete solution to the symmetric four root case. We conclude the chapter by giving several examples to illustrate our results. In Chapter 5, we give the relations for the symmetric five, six, and seven root cases. Using the relations for the general three and four root cases, we give, without proof, the suggested pattern for the relations for the general N root case. Then we give several examples of these nice polynomials we have found with a computer search, including the first known examples of nice symmetric or antisymmetric polynomials with five distinct roots and the first known examples of nice polynomials with six distinct roots. We conclude this thesis by presenting several open problems and new conjectures suggested by our results, examples, and computer searches.

CHAPTER 2

NICE SYMMETRIC AND ANTISYMMETRIC POLYNOMIALS WITH AN ODD NUMBER OF ROOTS

2.1. Introduction

Previous Results. In 1990, Chris Caldwell [**3**] published the first known paper on nice symmetric polynomials. In his paper, he gives an explicit formula [**3**, p. 37] for all nice symmetric quartics and the first five examples of nice nonsymmetric quartics with four distinct roots. See the introduction to Chapter 4 for more information on his results. The first result on nice antisymmetric polynomials appears in [**5**, Theorem 5.1].

Chapter Summary. In this chapter, we present several important results about nice symmetric or antisymmetric polynomials with an odd number of roots. Although these results are not complete, they have dramatically helped reduce the amount of work of a computer search for examples. We conclude the chapter with several open problems and conjectures suggested by our results and computer searches.

2.2. New Results

We first consider the case where $p(x) \in \mathbb{Z}[x]$ is a nice symmetric or antisymmetric polynomial with an odd number of distinct roots. We call a nice polynomial $p(x)$ *symmetric* if there exists a number c called the *center* such that $p(c-x) = p(c+x)$

for all real x and *antisymmetric* if $p(c-x) = -p(c+x)$ for all real x. Since the average of the roots of nice polynomials, which is the center of the set of roots of nice symmetric or antisymmetric polynomials, is an integer [**5**, Proposition 4.1], we may center $p(x)$ at the origin. Because $p(x)$ has an odd number of distinct roots, the center is a root of multiplicity $m_0 > 0$.

Although the following lemma may seem trivial, we mention it because of its importance.

LEMMA 2.2.1. *Let $p(x)$ be a symmetric or antisymmetric polynomial of degree d with an odd number of roots, with the center as a root of multiplicity $m_0 > 0$. Then d and m_0 have the same parity.*

PROOF. The conclusion follows by the symmetry of the set of roots of $p(x)$. ■

Notation. If we center $p(x)$ at the origin, then m_0 denotes the multiplicity of the root 0, so $p(x)$ has $d - m_0$ nonzero roots. Since half of these are positive and half are negative, we may denote these roots by $\pm r_1, ..., \pm r_k$, where $k = (d - m_0)/2$. By Rolle's Theorem, $p(x)$ has $(d-1) - (m_0 - 1) = d - m_0$ nonzero critical points. For the same reason, we may denote these critical points by $\pm c_1, ..., \pm c_k$.

We now prove the following theorem, which is implied by [**5**, Theorem 5.1], with essentially the same proof. We extend [**5**, Theorem 5.1] to include nice symmetric polynomials with an odd number of roots.

THEOREM 2.2.2. *If $p(x)$ is a nice symmetric or antisymmetric polynomial of degree d whose center is a root of multiplicity $m_0 > 0$, then d/m_0 is the square of a rational number.*

PROOF. Assume $p(x)$ is centered at the origin, and let $\pm r_1, ..., \pm r_k$ denote the nonzero roots of $p(x)$ and $\pm c_1, ..., \pm c_k$ denote the nonzero critical points of $p(x)$, where $k = (d - m_0)/2$. Using the equation $m_0 S_{d-m_0}(\pm r_1, ..., \pm r_k) = d S_{d-m_0}(\pm c_1, ..., \pm c_k)$ from system (1.3.1), we have $m_0 r_1^2 \cdots r_k^2 = d c_1^2 \cdots c_k^2$, which implies that d/m_0 is the square of a rational number. ∎

The following corollary is a direct consequence of Theorem 2.2.2.

COROLLARY 2.2.3. *If $p(x)$ is a nice symmetric or antisymmetric polynomial with an odd number of roots whose degree d is a square, then the multiplicity of its root 0 is a square with the same parity as d.*

PROOF. By Theorem 2.2.2, d/m_0 is the square of a rational number. Therefore, if d is a square, then m_0 is also a square. By Lemma 2.2.1, d and m_0 have the same parity. ∎

Remark. Because the equation in the proof of Theorem 2.2.2 is only one of the equations from system (1.3.1), Theorem 2.2.2 gives necessary but not sufficient conditions for the existence of nontrivial nice symmetric and antisymmetric polynomials with an odd number of roots. Sufficient conditions for existence have not been determined. In fact, computer searches for certain types suggest that the conditions stated in Theorem 2.2.2 are not sufficient. See Conjecture 1 and Problem 1 at the end of this chapter.

We now derive a formula that gives the number $N(d)$ of m_0's such that d and m_0 have the same parity and $d/m_0 > 1$ is the square of a rational number. The following theorem gives the value for $N(d)$ in all cases.

THEOREM 2.2.4. *Let d be a positive integer, and let a^2 be the largest square that divides d. Let e be the largest exponent such that 2^e divides d. Then the number $N(d)$ takes the following values depending on the prime factorization of d:*

(a) *If d is odd, then*

$$N(d) = (a-1)/2. \tag{2.2.1}$$

(b) *If d and $e > 0$ are even, then*

$$N(d) = (a-2)/2. \tag{2.2.2}$$

(c) *If d is even and e is odd, then*

$$N(d) = a - 1. \tag{2.2.3}$$

PROOF. By definition of a, $d = a^2c$, where $c > 0$ has no repeated prime factor. Since d/m_0 is the square of a rational number, $m_0 = b^2c$. Since $0 < m_0 < d$, $b < a$.

(a) If d is odd, then a^2 is odd. Hence, by Lemma 2.2.1, m_0 is odd, which implies that b^2 is odd. Therefore, b can take any odd value from 1 to $a-2$, so $N(d) = (a-1)/2$.

(b) If d and $e > 0$ are even, then c is odd. Since $m_0 = b^2c$ is even, by Lemma 2.2.1, and c is odd, b^2 is even. Therefore, b can take any even value from 2 to $a-2$

if $a > 2$. If $a = 2$, then no such $m_0 < d$ can be found. In either case, we have $N(d) = (a-2)/2$.

(c) If d is even and e is odd, then c is even. Since $m_0 = b^2c$ and c are even, b^2 is either even or odd. Hence, b can take any value from 1 to $a-1$, so $N(d) = a-1$. ■

The following corollaries are direct consequences of Theorems 2.2.2 and 2.2.4.

COROLLARY 2.2.5. *If $p(x)$ is a nontrivial nice symmetric or antisymmetric polynomial of degree d with an odd number of distinct roots, then there are at most $N(d)$ possible values for the multiplicity m_0 of the center of the set of roots of $p(x)$.*

PROOF. This result follows immediately from Theorems 2.2.2 and 2.2.4. ■

COROLLARY 2.2.6. *Let $p(x)$ be a nontrivial nice symmetric or antisymmetric polynomial of degree d with an odd number of distinct roots. Let e be the largest exponent such that 2^e divides d. Then, the largest square that divides the degree d is*

(a) *greater than 1 if d is odd or if d is even and e is odd;*

(b) *greater than 4 if d is even and $e > 0$ is even.*

PROOF. By (2.2.1) and (2.2.3), $N(d) = 0$ iff $a^2 = 1$; and, by (2.2.2), $N(d) = 0$ iff $a^2 \leq 4$. ■

Remark. By Corollary 2.2.6, there exist no nontrivial nice symmetric or antisymmetric polynomials of degrees 3-7, 10-15, 17, 19-23, 26, and so on with an odd number of distinct roots.

Application. As a first application, Theorem 2.2.2 and Corollary 2.2.6 have dramatically decreased the amount of work of a computer search for examples. The reduction in work has allowed us to find the first known examples of nice antisymmetric and symmetric polynomials with more than three and four roots respectively:

$$p(x) = x(x^2 - 33^2)(x^2 - 513^2)^3, \tag{2.2.4}$$

$$p'(x) = 9(x^2 - 513^2)^2(x^2 - 297^2)(x^2 - 19^2);$$

and

$$p(x) = x^2(x^2 - 598^2)^2(x^2 - 238^2), \tag{2.2.5}$$

$$p'(x) = 8x(x^2 - 598^2)(x^2 - 161^2)(x^2 - 442^2).$$

Other inequivalent examples with five roots are given in Chapter 5.

Comments on Table 2.3.1. Table 2.3.1 at the end of this chapter classifies all nontrivial nice symmetric or antisymmetric polynomials of degree 26 or less with an odd number of roots. Note that this table does not include nice symmetric or antisymmetric polynomials with three roots. See Table 3.3.1 at the end of Chapter 3 for the classification of nice symmetric or antisymmetric polynomials with three roots. The following notation and definitions are used in all our tables.

Notation. We say that a polynomial is of the type $(m_1, m_2, ..., m_k)$ if the polynomial has distinct roots of multiplicity $m_1 \geq m_2 \geq ... \geq m_k$ without regard to the size of the roots. For example, the polynomials $p_1(x) = x(x-3)^2(x+4)^5(x-1)$ and $p_2(x) = x^2(x+1)(x-5)(x-1)^5$ are of the type (5,2,1,1).

Definitions. If $p(x)$ is a nice polynomial with roots $r_1 \leq r_2 \leq ... \leq r_d$, then the difference $r_d - r_1$ is called the *diameter of the set of roots of* $p(x)$. Half this distance is the *radius*. The *smallest example* of a given type is the example with the smallest diameter for its set of roots.

2.3. Open Problems and Conjectures

CONJECTURE 1. Theorem 2.2.2 does not give sufficient conditions for the existence of nontrivial nice symmetric or antisymmetric polynomials with an odd number of roots.

CONJECTURE 2. There exists an odd $N > 3$ such that formulas for all nice symmetric or antisymmetric polynomials with N, $N + 2$, $N + 4$,... distinct roots cannot be found. In other words, it is impossible to find a formula for all nice symmetric or antisymmetric polynomials with an odd number of roots.

PROBLEM 1. Prove or disprove Conjecture 1 above.

PROBLEM 2. Prove or disprove Conjecture 2 above.

PROBLEM 3. Find a nice symmetric or antisymmetric polynomial with seven roots.

PROBLEM 4. Are the examples given in (2.2.4) and (2.2.5) the only inequivalent examples of those types?

Classification of Nontrivial Nice Symmetric and Antisymmetric Polynomials with an Odd Number of Roots						
m_0 = List of all m_0's satisfying Theorem 2.2.2 N_{classes} = Number of equivalence classes N_{examples} = Number of known examples r_{min} = Radius of the smallest example S = Solved U = Unsolved or Unknown						
Degree	m_0	Type	Status	N_{classes}	N_{examples}	r_{min}
3-7		All types	S	0 by Thm. 2.2.2		
8	2	(2,2,2,1,1)	U	U	1 in (2.2.5)	598
8	2	(2,1,1,1,1,1,1)	U	U	0	
9	1	(2,2,2,2,1)	U	U	0	
9	1	(3,3,1,1,1)	U	U	1 in (2.2.4)	513
9	1	All others	U	U	0	
10-15		All types	S	0 by Thm. 2.2.2		
16	4	All types	U	U	0	
17		All types	S	0 by Thm. 2.2.2		
18	2,8	All types	U	U	0	
19-23		All types	S	0 by Thm. 2.2.2		
24	6	All types	U	U	0	
25	1,9	(11,11,1,1,1)	U	U	2	385 in (3.2.5)
25	1,9	All others	U	U	0	
26		All types	S	0 by Thm. 2.2.2		

TABLE 2.3.1. Classification Table

CHAPTER 3

NICE POLYNOMIALS WITH THREE ROOTS

3.1. Introduction

Previous Results. The earliest work on nice polynomials focused on nice cubics [**1**], [**4**], and [**7**]. These authors had solved the cubic case by using Pythagorean triples. The formula they give is as follows:

$$p(x) = x[x - 3m(m + 2n)][x - 3n(2m + n)], \tag{3.1.1}$$

$$p'(x) = 3(x - 3mn)[x - (m + 2n)(2m + n)],$$

where $m, n \in \mathbb{Z}$ are relatively prime.

Evard [**5**] solved two other special cases with three roots using Pythagorean triples. He gives the following two formulas [**5**, Formula A10] for nice quintics with a triple root at the origin:

$$p(x) = x^3[x - 5m(4n - 5m)][x - 5n(3n - 4m)], \tag{3.1.2}$$

$$p'(x) = 5x^2[x - 5m(3n - 4m)][x - 3n(4n - 5m)];$$

and

$$p(x) = x^3[x - 5m(m - 4n)][x - 5n(4m - 15n)], \tag{3.1.3}$$

$$p'(x) = 5x^2[x - 15n(m - 4n)][x - m(4m - 15n)],$$

where $m, n \in \mathbb{Z}$ are relatively prime.

Chapter Summary. In this chapter, we present the complete solution to the symmetric three root case. (Note that, for simplicity, we favor this expression over the more complicated expression *antisymmetric or symmetric three root case.*) The complete solution of a certain case consists of a formula for all nice (or Q-nice) polynomials of that specific case, necessary and sufficient conditions for existence, and the number of equivalence classes for that specific case. We then present the complete solution to the general three root case, and we note the differences between the symmetric and general three root cases. To conclude this chapter, we present several examples to illustrate our results.

3.2. Symmetric Three Root Case

To begin, we consider the special case where $p(x) \in \mathbb{Z}[x]$ is a nice symmetric or antisymmetric polynomial with three distinct roots. If we center $p(x)$ at the origin, then $p(x)$ has the form $p(x) = x^{m_0}(x^2 - r^2)^k$, where $k = (d - m_0)/2$; and the derivative has the form $p'(x) = dx^{m_0-1}(x^2 - r^2)^{k-1}(x^2 - c^2)$.

We first need the relations between the roots and critical points of all polynomials $p(x)$ with three roots, then we may use these relations to solve the symmetric and general three root cases. We may shift $p(x)$ so that one of its roots is 0. If we do so, then $p(x)$ has the form $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}$; and the derivative has the form $p'(x) = dx^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - c_1)(x - c_2)$. The following lemma gives the relations between the roots and critical points for all polynomials with three roots.

LEMMA 3.2.1. *Let* $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}$ *be any polynomial of degree* d *with integer coefficients and three integer roots. Then* $p(x)$ *is nice iff there exist integers* c_1 *and* c_2 *such that*

$$(d - m_1)r_1 + (d - m_2)r_2 = d(c_1 + c_2), \tag{3.2.1}$$

$$m_0 r_1 r_2 = d c_1 c_2. \tag{3.2.2}$$

PROOF. This result is proven in [**5**, Theorem 7.1]. ■

If $p(x)$ is symmetric or antisymmetric, then the relations (3.2.1) and (3.2.2) reduce as follows:

LEMMA 3.2.2. *Let* $p(x) = x^{m_0}(x^2 - r^2)^k$ *be a symmetric or antisymmetric polynomial of degree* d *with integer coefficients and three integer roots. Then* $p(x)$ *is nice iff there exists an integer* c *such that*

$$m_0 r^2 = dc^2. \tag{3.2.3}$$

PROOF. If we let $r_1 = -r_2 = r$, $c_1 = -c_2 = c$, and $m_1 = m_2$, then (3.2.1) becomes 0=0, and (3.2.2) becomes $m_0 r^2 = dc^2$, as needed. ■

The following theorem gives the complete solution to the symmetric three root case.

THEOREM 3.2.3. *There exists a nice symmetric or antisymmetric polynomial* $p(x)$ *of degree* d *with three roots whose center is a root of multiplicity* $m_0 > 0$ *iff* $d/m_0 > 1$ *is the square of a rational number and* d *and* m_0 *have the same parity. For every* d *and* m_0 *satisfying this condition, there exists only one equivalence class* *of such nice polynomials, and the representative for this class is given by*

$$p(x) = x^{m_0}(x^2 - s^2)^k, \tag{3.2.4}$$

$$p'(x) = dx^{m_0-1}(x^2 - s^2)^{k-1}(x^2 - t^2),$$

where $\frac{d}{m_0} = \frac{s^2}{t^2}$, *with* s *and* t *relatively prime.*

PROOF. To prove the first part, we note that the equation $m_0 r^2 = dc^2$ has nonzero integer solutions where $r^2 \neq c^2$ iff $d/m_0 > 1$ is the square of a rational number. By Lemma 2.2.1, d and m_0 have the same parity.

To prove the second part, we need to show that, for any m_0 that gives a natural number solution to (3.2.3) where r and c are relatively prime, the solution is unique. By (3.2.3), $r^2 = \frac{d}{m_0}c^2 = \frac{s^2}{t^2}c^2$ where s^2 and t^2 are perfect squares with no common factor. Since $r = \frac{s}{t}c$ and$r$ and c are natural numbers, $c = tc_0$ and $r = sc_0$ for some natural number c_0. Since r and c are relatively prime, $c_0 = 1$. We conclude that $r = s$ and $c = t$ is the only natural number solution to (3.2.3) where r and c are relatively prime. ■

Remarks. (1). Note that, in this special case, the requirement that $d/m_0 > 1$ be the square of a rational number is a necessary and sufficient condition for existence since the single equation (3.2.3) is equivalent to the system (1.3.1). Furthermore, by Theorem 3.2.3, for all degrees d, there exist exactly $N(d)$ equivalence classes of nice symmetric or antisymmetric polynomials with three roots.

(2). A formula for all nice antisymmetric polynomials of odd square degree with three roots is found in [**5**, Formula A2].

(3). Although every type of nice symmetric or antisymmetric polynomial with three roots that exists is unique, not every type with five roots that exists is unique. The following two examples prove this rather surprising fact:

$$p(x) = x(x^2 - 65^2)(x^2 - 385^2)^{11}, \tag{3.2.5}$$

$$p'(x) = 25(x^2 - 385^2)^{10}(x^2 - 35^2)(x^2 - 143^2);$$

$$p(x) = x(x^2 - 115^2)(x^2 - 1235^2)^{11}, \tag{3.2.6}$$

$$p'(x) = 25(x^2 - 1235^2)^{10}(x^2 - 65^2)(x^2 - 437^2).$$

Comments on Table 3.3.1. Table 3.3.1 classifies all nice symmetric or antisymmetric polynomials of degree 26 or less with three roots. See the end of the chapter for this table.

3.3. General Three Root Case

Now we consider all Q-nice polynomials $p(x) \in \mathbb{Q}[x]$ with three roots. To solve the general three root case, we find and solve a system of equations relating the roots and critical points for all polynomials with three roots (see (3.2.1) and (3.2.2) above). Recall that $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}$ and $p'(x) = dx^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - c_1)(x - c_2)$.

The following algebraic theorem is a generalization of our main result for this chapter (Theorem 3.3.2).

THEOREM 3.3.1. *All rational solutions to the system*

$$a_1x + a_2y = b_1z + b_2w, \tag{3.3.1}$$

$$c_1xy = c_2zw \tag{3.3.2}$$

where $a_1, a_2, b_1, b_2, c_1,$ *and* c_2 *are nonzero rational numbers are given by*

$$w = \frac{c_1x(a_1x - b_1z)}{b_2c_1x - a_2c_2z} \tag{3.3.3}$$

and

$$y = \frac{c_2z(a_1x - b_1z)}{b_2c_1x - a_2c_2z} \tag{3.3.4}$$

for any rational numbers x *and* z *such that* $b_2c_1x - a_2c_2z \neq 0$.

PROOF. An easy substitution shows that (3.3.3) and (3.3.4) are solutions to (3.3.1) and (3.3.2). Because x and z are free variables for the system (3.3.1)-(3.3.2), all rational solutions to (3.3.1)-(3.3.2) are given by (3.3.3)-(3.3.4). ■

Remark. We note that, if we solve (3.3.1) and (3.3.2) for x and y or for z and w, we obtain formulas where rational solutions are not easily found. A similar situation occurs for the system (3.2.1) and (3.2.2).

The following theorem, our main result for this chapter, gives a formula for all Q-nice polynomials with three roots.

THEOREM 3.3.2. *The polynomial* $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}$ *is Q-nice with* $p'(x) = dx^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - c_1)(x - c_2)$ *iff*

$$r_1 = \frac{dc_2(r_2 - c_2) - c_2r_2m_2}{c_2m_1 + r_2m_0 - dc_2} \tag{3.3.5}$$

and

$$c_1 = \frac{r_1r_2m_0}{dc_2} \tag{3.3.6}$$

for any nonzero rational numbers r_2 and c_2 such that $c_2m_1 + r_2m_0 - dc_2 \neq 0$.

PROOF. Since this theorem is a special case of Theorem 3.3.1, the conclusion follows. ■

Given the degree of $p(x)$ and the multiplicities of the three roots 0, r_1, and r_2—we choose nonzero r_2 and c_2 and use (3.3.5) and (3.3.6) above to find r_1 and c_1. Since division by zero is undefined, we must avoid zero denominators; this implies that $c_2m_1+r_2m_0-dc_2 \neq 0$. To guarantee this, we note that $c_2m_1+r_2m_0-dc_2 = 0$ implies $c_2m_1+r_2m_0 = dc_2$. Using $d = m_0+m_1+m_2$, we now have $c_2m_1+r_2m_0 = (m_0+m_1+m_2)c_2$, or $r_2m_0 = m_0c_2 + m_2c_2$. Dividing by $c_2 \neq 0$ gives us $(r_2/c_2)m_0 - m_0 = m_2$, which is false if r_2/c_2 is not an integer. Therefore, if we choose r_2 and c_2 such that r_2/c_2 is not an integer, we avoid the zero denominator in (3.3.5).

Finally, $r_1 \neq 0$, so we choose nonzero r_2 and c_2 such that $dc_2(r_2-c_2)-c_2r_2m_2 \neq 0$. Note that $dc_2(r_2-c_2) - c_2r_2m_2 = 0$ implies $dc_2r_2 - dc_2^2 = c_2r_2m_2$. Dividing both sides by $r_2c_2 \neq 0$ gives us $d - (c_2/r_2)d = m_2$, which is false if c_2/r_2 is not an integer (or if $c_2/r_2 > 1$). Therefore, if we choose r_2 and c_2 such that r_2/c_2 and c_2/r_2 are not integers, then (3.3.5) and (3.3.6) give nonzero values for r_1 and c_1.

Remark. Although choosing r_2 and c_2 such that r_2/c_2 and c_2/r_2 are not integers guarantees nonzero solutions to (3.3.5) and (3.3.6), other choices of r_2 and c_2 give nonzero solutions to (3.3.5) and (3.3.6). For example, we showed in the previous paragraph that if $c_2/r_2 > 1$, then $r_1 \neq 0$. In this case, r_2/c_2 is not an integer, so we avoid the zero denominator in (3.3.5). Therefore, we may also choose r_2 and c_2 so that c_2/r_2 is an integer greater than 1.

Theorem 3.3.2 suggests that all types of Q-nice polynomials with three roots exist. The following theorem proves this fact.

THEOREM 3.3.3. *All types of Q-nice polynomials with three roots exist.*

PROOF. As discussed above, we are guaranteed that r_1 and c_1 exist and are nonzero whenever r_2/c_2 and c_2/r_2 are not integers, so let $r_2 = 1$ and $c_2 = k$ where k is not an integer. Since $m_0 r_2 \neq dc_2$, then, by (3.3.6) $r_1 \neq c_1$. Furthermore, if $r_1 = r_2 = 1$, then, by (3.3.5), $2(d + m_2) = 2d - m_0 - 2m_1$, or $2m_2 + 2m_1 + m_0 = 0$, a contradiction. Hence, $r_1 \neq r_2 = 1$. Finally, note that $r_1 \neq c_2 = k$. If $r_1 = c_2 = k$, then, by (3.3.5), $dk(1-k) - km_2 = k(km_1 + m_0 - dk)$, or $d = m_0 + km_1 + m_2$, a contradiction since $k \neq 1$. By Rolle's Theorem, c_1 and c_2 are distinct and differ from the roots. Since m_0, m_1, and m_2 are arbitrary, we conclude that all types of Q-nice polynomials with three roots exist. ∎

Remarks. (1). Since k in the proof above is arbitrary, there exist infinitely many choices for r_2 and c_2 that give examples with exactly three roots.

(2). As a direct consequence of Theorem 3.2.3, there are no restrictions on the multiplicities m_0, m_1, and m_2 of the roots of Q-nice polynomials with three roots.

Before we count the number of equivalence classes of Q-nice polynomials with three roots, we make the following observations. Let $r_1, ..., r_d$ and $c_1, ..., c_{d-1}$ denote the roots and critical points of a Q-nice polynomial, and let $r'_1, ..., r'_d$ and $c'_1, ..., c'_{d-1}$ denote the corresponding roots and corresponding critical points of an equivalent Q-nice polynomial. Reflections, stretches, shrinks, and multiplications by scalars preserve ratios between corresponding roots and corresponding critical points; that is, $r_i/r_j = r'_i/r'_j$, $c_i/c_j = c'_i/c'_j$, and $r_i/c_j = r'_i/c'_j$ for all i and j. Horizontal shifts preserve distances between corresponding roots and corresponding critical points; that is, $r_i - r_j = r'_i - r'_j$, $c_i - c_j = c'_i - c'_j$, and $r_i - c_j = r'_i - c'_j$ for all i and j. Note that the power transformation preserves the union of the set of roots and the set of critical points of $p(x)$.

The following theorem gives us the number of equivalence classes of a given type of Q-nice polynomial with three roots.

THEOREM 3.3.4. *There exist infinitely many equivalence classes of a given type of Q-nice polynomial with three roots.*

PROOF. Transformations preserve ratios and differences as described above. Since r_2 and c_2 are free variables for solution (3.3.5) and (3.3.6), then, by the proof of Theorem 3.2.3, we may choose infinitely many values of r_2, c_2, r'_2, and c'_2 such that $r_2 = r'_2$ and $c_2 \neq c'_2$. Therefore, $\frac{r_2}{c_2} \neq \frac{r'_2}{c'_2}$ and $r_2 - c_2 \neq r'_2 - c'_2$; hence, the examples produced by solutions (3.3.5) and (3.3.6) are inequivalent. We, therefore, conclude that infinitely many equivalence classes exist for every type. ■

Remark. Note the remarkable differences between the symmetric three root case and the general three root case. Not all types of nice symmetric or antisymmetric polynomials with three roots exist, but all non-symmetric types with three roots exist. Furthermore, for the general case, we have infinitely many equivalence classes for each type; but, for the symmetric three root case, we have at most one equivalence class for each type.

The following examples are nice polynomials we have found using (3.3.5) and (3.3.6). For these examples, the type is specified by (m_0, m_1, m_2).

EXAMPLE 1. If we specify the type (7,3,2) and we choose $r_2 = 20$ and $c_2 = 15$, then we have that $r_1 = 60$ and $c_1 = 140/3$. This polynomial is equivalent to $p(x) = x^7(x-36)^3(x-12)^2$ with derivative $p'(x) = 12x^6(x-36)^2(x-12)(x-28)(x-9)$.

EXAMPLE 2. If we specify the type (3,2,4) and we choose $r_2 = 7$ and $c_2 = 5$, then we have that $r_1 = -5$ and $c_1 = -7/3$. This polynomial is equivalent to $p(x) = x^3(x+15)^4(x-21)^2$ with derivative $p'(x) = 9x^2(x+15)^3(x-21)(x+7)(x-15)$.

EXAMPLE 3. We can show that (3.3.5) and (3.3.6) give examples of nice symmetric and antisymmetric polynomials with three roots. Using (3.2.3), if we let $c_2 = c$, $r_2 = \sqrt{\frac{dc^2}{m_0}}$, $m_1 = m_2 = k$, then we have that $r_1 = -r_2$ and $c_1 = -c$, as needed.

We end with an example that illustates Theorem 3.3.4.

EXAMPLE 4. Let $p_1(x)$ and $p_2(x)$ be of type (5,4,3). Choose $r_2 = r_2' = 4$, $c_2 = 3$, and $c_2' = 1$. By (3.3.5) and (3.3.6), $r_1 = 15$, $r_1' = 5/3$, $c_1 = 25/3$, and $c_1' = 25$. Note that the first polynomial is equivalent to $p_1(x) = x^5(x-45)^4(x-12)^3$ with $p_1'(x) = 12x^4(x-45)^3(x-12)^2(x-25)(x-9)$, and the second is equivalent to $p_2(x) = x^5(x-5)^4(x-12)^3$ with $p_2'(x) = 12x^4(x-5)^3(x-12)^2(x-75)(x-3)$. Note that these polynomials are inequivalent examples of the type (5,4,3).

Classification of Nontrivial Nice Symmetric and Antisymmetric Polynomials with Three Roots			
			N_{classes} = Number of equivalence classes $r_{\min}$ = Radius of the smallest example
Degree	Type	N_{classes}	$r_{\min}$
3-7	All types	0	
8	(3,3,2)	1	2
8	All others	0	
9	(4,4,1)	1	3
9	All others	0	
10-15	All types	0	
16	(6,6,4)	1	2
16	All others	0	
17	All types	0	
18	(8,8,2)	1	3
18	(8,5,5)	1	3
19-23	All types	0	
24	(9,9,6)	1	2
24	All others	0	
25	(12,12,1)	1	5
25	(9,8,8)	1	5
25	All others	0	
26	All types	0	

TABLE 3.3.1. Classification Table

CHAPTER 4

NICE POLYNOMIALS WITH FOUR ROOTS

4.1. Introduction

Previous Results. Chris Caldwell [**3**] solved the symmetric quartic case in 1990. He gives the following formula:

$$p(x) = [x^2 - (n^2 - m^2 - 2mn)^2][x^2 - (n^2 - m^2 + 2mn)^2], \tag{4.1.1}$$
$$p'(x) = 4x[x^2 - (m^2 + n^2)^2],$$

where $m, n \in \mathbb{Z}$ are relatively prime. He also gives the first five known examples of nice nonsymmetric quartics with four distinct roots, the first one being

$$p(x) = (x + 9817)(x + 1307)(x - 2741)(x - 8383), \tag{4.1.2}$$
$$p'(x) = 4(x + 6931)(x - 648)(x - 6283).$$

Evard had found 358 other examples of nice non-symmetric quartics, the smallest one [**5**, Example B6] being

$$p(x) = x(x - 50)(x - 176)(x - 330), \tag{4.1.3}$$
$$p'(x) = 4(x - 22)(x - 120)(x - 275).$$

Buchholz and MacDougall have shown that nontrivial quartics of the types (2,1,1) and (3,1) are totally nice whereas the type (2,2) is not. And they have conjectured that quartics of the type (1,1,1,1) are not totally nice [**2**, Table 1].

Chapter Summary. In this chapter, we give the relations between the roots and critical points for all polynomials with four roots. Then we present the complete solution to the symmetric four root case, giving several examples to illustrate our results. We conclude with some open problems.

4.2. General Four Root Case

To begin, we consider all nice polynomials $p(x) \in \mathbb{Z}[x]$ with four distinct roots. If we shift $p(x)$ so that one root is at zero, then $p(x)$ has the form $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3}$; and the derivative has the form $p'(x) = dx^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - r_3)^{m_3-1}(x - c_1)(x - c_2)(x - c_3)$. The following lemma gives the relations between the roots and critical points of all polynomials with four roots.

LEMMA 4.2.1. *A polynomial $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$ of degree d with integer coefficients and with four integer roots is nice iff there exist integers c_1, c_2, and c_3 such that*

$$(d - m_1)r_1 + (d - m_2)r_2 + (d - m_3)r_3 = d(c_1 + c_2 + c_3), \tag{4.2.1}$$

$$(m_0 + m_3)r_1r_2 + (m_0 + m_2)r_1r_3 + (m_0 + m_1)r_2r_3 = d(c_1c_2 + c_1c_3 + c_2c_3), \tag{4.2.2}$$

$$m_0r_1r_2r_3 = dc_1c_2c_3. \tag{4.2.3}$$

PROOF. Differentiating $p(x)$, we have

$$p'(x) = q(x)x^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - r_3)^{m_3-1},$$

where $q(x) = m_0(x - r_1)(x - r_2)(x - r_3) + m_1x(x - r_2)(x - r_3) + m_2x(x - r_1)(x - r_3) + m_3x(x - r_1)(x - r_2)$. As mentioned above, the derivative of $p(x)$ has the form $p'(x) = dx^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1}(x-c_1)(x-c_2)(x-c_3)$. Therefore,

$q(x) = d(x - c_1)(x - c_2)(x - c_3)$. By expanding both forms of $q(x)$ and equating coefficients, we obtain the relations above. ■

Application. Although the general four root case has not been solved (see Problems 5-7 at the end of this chapter), the relations above can help reduce the amount of work of a computer search for examples. Furthermore, if it is possible to solve the general four root case, the relations above can help us do so.

4.3. Symmetric Four Root Case

We now consider the case where $p(x) \in \mathbb{Q}[x]$ is a Q-nice symmetric polynomial with four roots. If we center $p(x)$ at the origin, then $p(x)$ has the form $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$; and the derivative has the form $p'(x) = dx(x^2 - r_1^2)^{m_1 - 1}(x^2 - r_2^2)^{m_2 - 1}(x^2 - c^2)$. The following lemma gives the relations between the roots and critical points of all symmetric polynomials with four distinct roots.

LEMMA 4.3.1. *A symmetric polynomial $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ of degree d with integer coefficients and with four integer roots is nice iff there exists an integer c such that*

$$m_2 r_1^2 + m_1 r_2^2 = (m_1 + m_2)c^2. \tag{4.3.1}$$

PROOF. Differentiating $p(x)$, we obtain

$$p'(x) = 2x(x^2 - r_1^2)^{m_1 - 1}(x^2 - r_2^2)^{m_2 - 1}[(m_1 + m_2)x^2 - (m_2 r_1^2 + m_1 r_2^2)].$$

By Rolle's Theorem, there exist nonzero critical points $\pm c$ of $p(x)$ which differ from the roots. Hence, the critical points $\pm c$ satisfy the equation $(m_1 + m_2)c^2 - (m_2 r_1^2 + m_1 r_2^2) = 0$, which is the relation stated above. ■

Remark. We cannot derive the relations for the symmetric four root case by using the relations for the general four root case because, in the symmetric four root case, $p(x)$ does not have a root at zero if we center $p(x)$ at the origin. Therefore, the relations for the general four root case cannot be used to derive the relations for the symmetric four root case.

The following theorem gives a formula for all Q-nice symmetric polynomials with four roots. To find a formula, we solve (4.3.1) for all rational numbers r_1, r_2, and c.

THEOREM 4.3.2. *The symmetric polynomial* $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ *with* $p'(x) = dx(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - c^2)$ *is Q-nice iff*

$$c = \frac{m_2a^2 + m_1b^2}{2m_2a - 2m_1b}, \tag{4.3.2}$$

$$r_1 = c - a, \tag{4.3.3}$$

$$r_2 = c + b \tag{4.3.4}$$

for some natural numbers a *and* b *where* $2m_2a - 2m_1b \neq 0$.

PROOF. Assume r_1, r_2, $c > 0$ with $r_1 < r_2$. Since $r_1 < c < r_2$, $c = r_1 + a$ and $c = r_2 - b$ for some natural numbers a and b. Substituting $c - a$ for r_1 and $c + b$ for r_2 in (4.3.1), we have $m_2(c^2 - 2ac + a^2) + m_1(c^2 + 2bc + b^2) = (m_1 + m_2)c^2$, which simplifies to $m_2a^2 + m_1b^2 = (2m_2a - 2m_1b)c$. Dividing both sides by $2m_2a - 2m_1b$, we obtain (4.3.2) above. Equations (4.3.3) and (4.3.4) follow from the definition of a and b. ■

Remarks. (1). The requirement that $2m_2a - 2m_1b \neq 0$ is equivalent to $a \neq (\frac{m_1}{m_2})b$.

(2). If a and b are natural numbers, then, by (4.3.3) and (4.3.4), $r_1 \neq c$ and $r_2 \neq c$. Hence, to find all values for a and b where r_1^2, r_2^2, and c^2 are not distinct, all we need are the values for natural numbers a and b where $r_1 = \pm r_2$. Solving the equation $r_1 = \pm r_2$ for a and b, we have that $a = -b$, which is false for all natural numbers a and b. Therefore, r_1^2, r_2^2, and c^2 are all distinct whenever $a \neq (\frac{m_1}{m_2})b$.

Theorem 4.3.2 suggests that all types of Q-nice symmetric polynomials with four roots exist. The following theorem proves this fact.

THEOREM 4.3.3. *All types of Q-nice symmetric polynomials with four roots exist.*

PROOF. As discussed above, r_1^2, r_2^2, and c^2 are distinct whenever $a \neq (\frac{m_1}{m_2})b$. Since we can always choose natural numbers a and b whenever $a \neq (\frac{m_1}{m_2})b$ for any m_1 and m_2, we conclude that all types of Q-nice symmetric polynomials with four roots exist. ∎

Remarks. (1). As a direct consequence of Theorem 4.3.3, there are no restrictions on the multiplicities m_1 and m_2 of the roots of Q-nice symmetric polynomials with four roots.

(2). The values a and b given in Formulas (4.3.2)-(4.3.4) can be interpreted geometrically as follows: The number a gives the distance between the critical point c and the root r_1, and the number b gives the distance between c and the root r_2. If we were to multiply (4.3.2), (4.3.3), and (4.3.4) by $2m_2a - 2m_1b$ to clear fractions, then a and b could no longer be interpreted geometrically as mentioned above. For this reason, we prefer the rational forms of (4.3.2)-(4.3.4) given above.

The following theorem states the number of equivalence classes of Q-nice symmetric polynomials with four roots.

THEOREM 4.3.4. *Given a type of Q-nice symmetric polynomial with four roots, there exist infinitely many equivalence classes for that type.*

PROOF. Since all examples $p(x)$ found by our formula are monic and since $p(x)$ is an even function, unequal examples found by our formula are equivalent under stretches or shrinks. Since stretches and shrinks preserve the ratio a/b, we can choose infinitely many different ratios a/b for any given type; hence, for any given type, infinitely many equivalence classes exist. ■

We now give several examples we have found using (4.3.2)-(4.3.4).

EXAMPLE 5. If we specify the type (3,3,1,1) and choose $a = 1$ and $b = 5$, then our formulas give $r_1 = -26/7$, $r_2 = 16/7$, and $c = -19/7$ if $m_1 = 3$ and $m_2 = 1$. Note that this example is equivalent to $p(x) = (x^2 - 26^2)^3(x^2 - 16^2)$ with derivative $p'(x) = 8x(x^2 - 26^2)^2(x^2 - 19^2)$. If $m_1 = 1$ and $m_2 = 3$, then our formulas give $r_1 = -8$, $r_2 = -2$, and $c = -7$. Note that this example is equivalent to $p(x) = (x^2 - 2^2)^3(x^2 - 8^2)$ with derivative $p'(x) = 8x(x^2 - 2^2)^2(x^2 - 7^2)$.

EXAMPLE 6. If we specify the type (7,7,5,5) and choose $a = 2$ and $b = 7$, then our formulas give $r_1 = -173/26$, $r_2 = 61/26$, and $c = -121/26$ if $m_1 = 7$ and $m_2 = 5$. Note that this example is equivalent to $p(x) = (x^2 - 173^2)^7(x^2 - 61^2)^5$ with derivative $p'(x) = 24x(x^2 - 173^2)^6(x^2 - 61^2)^4(x^2 - 121^2)$. If $m_1 = 5$ and $m_2 = 7$, then our formulas give $r_1 = -17/2$, $r_2 = 1/2$, and $c = -13/2$. Note that this example is equivalent to $p(x) = (x^2 - 17^2)^5(x^2 - 1)^7$ with derivative $p'(x) = 24x(x^2 - 17^2)^4(x^2 - 1)^6(x^2 - 13^2)$.

The following example illustrates Theorem 4.3.4.

EXAMPLE 7. If we specify the type (3,3,2,2) and choose $a = 1$ and $b = 4$, then our formulas give $r_1 = -7/2$, $r_2 = 3/2$, and $c = -5/2$ if $m_1 = 3$ and $m_2 = 2$. Note that this example is equivalent to $p_1(x) = (x^2 - 7^2)^3(x^2 - 3^2)^2$, $p_1'(x) = 10x(x^2 - 7^2)^2(x^2 - 3^2)(x^2 - 5^2)$. If we choose $a = 1$ and $b = 2$, then our formulas give $r_1 = -11/4$, $r_2 = 1/4$, and $c = -7/4$. Note that this example is equivalent to $p_2(x) = (x^2 - 11^2)^3(x^2 - 1)^2$, $p_2'(x) = 10x(x^2 - 11^2)^2(x^2 - 1)(x^2 - 7^2)$. Note that both these examples are inequivalent examples of the type (3,3,2,2).

4.4. Open Problems

PROBLEM 5. Find a formula for all nice polynomials with four roots.

PROBLEM 6. Which types of nice polynomials with four roots exist?

PROBLEM 7. Given a type of nice polynomial with four roots, how many equivalence classes exist for that type?

CHAPTER 5

NICE POLYNOMIALS WITH FIVE OR MORE ROOTS

5.1. Introduction

Previous Results. So far, no examples of nice polynomials with five or six roots have been published. However, in his submitted paper, Evard gives a nice quintic with five distinct real roots [**5**, Example B7], the only known example so far:

$$\begin{aligned} p(x) &= x(x-180)(x-285)(x-460)(x-780), \\ p'(x) &= 5(x-60)(x-230)(x-390)(x-684). \end{aligned} \tag{5.1.1}$$

He also gives two nice quintics with five distinct roots in $\mathbb{G}$, the Gaussian integers [**5**, Example B8]:

$$\begin{aligned} p(x) &= x(x-595)(x-1020)[(x-220)^2+40,000], \\ p'(x) &= 5(x-170)^2(x-420)(x-884); \end{aligned} \tag{5.1.2}$$

and

$$\begin{aligned} p(x) &= x(x-585)(x-1040)[(x-270)^2+44,100], \\ p'(x) &= 5(x-130)(x-312)(x-390)(x-900). \end{aligned} \tag{5.1.3}$$

Buchholz and MacDougall have shown that nice quintics of the type (2,2,1) are not totally nice [**2**, Theorem 1] and that the type (3,2) is not totally nice [**2**, Table 2]. However, they have shown that the type (4,1) is totally nice [**2**, Table

2]. Furthermore, they have conjectured that the types (1,1,1,1,1), (2,1,1,1), (3,1,1) are not totally nice. In other words, if their conjectures are correct, then the only nontrivial nice quintics that are totally nice are of the type (4,1).

Chapter Summary. In this chapter, we present several results about nice symmetric or antisymmetric polynomials with five, six, or seven roots. Each of these results can help reduce the amount of work of a computer search for examples. We also give, without proof, the suggested pattern for the relations between the roots and critical points of all nice polynomials with N roots. We then give several examples we have found using our results, including the first known examples of nice symmetric or antisymmetric polynomials with five distinct roots and the first known examples of nice polynomials with six distinct roots. We conclude with several new conjectures and open problems suggested by our results, examples, and computer searches.

5.2. Symmetric Five Root Case

We first consider the case where $p(x) \in \mathbb{Z}[x]$ is a nice symmetric or antisymmetric polynomial with five distinct roots. If we center $p(x)$ at the origin, then $p(x)$ has the form $p(x) = x^{m_0}(x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$; and the derivative has the form $p'(x) = dx^{m_0-1}(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - c_1^2)(x^2 - c_2^2)$. The following lemma gives the relations between the roots and critical points of all symmetric and antisymmetric polynomials with five roots.

LEMMA 5.2.1. *Suppose* $p(x) = x^{m_0}(x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ *is a symmetric or antisymmetric polynomial with integer coefficients and with five integer roots. Then*

$p(x)$ is nice iff there exist integers c_1 and c_2 such that

$$(m_0 + 2m_2)r_1^2 + (m_0 + 2m_1)r_2^2 = d(c_1^2 + c_2^2), \tag{5.2.1}$$

$$m_0 r_1^2 r_2^2 = d c_1^2 c_2^2. \tag{5.2.2}$$

PROOF. Differentiating $p(x)$, we have

$$p'(x) = q(x)x^{m_0-1}(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}$$

where $q(x) = m_0(x^2 - r_1^2)(x^2 - r_2^2) + 2m_1x^2(x^2 - r_2^2) + 2m_2x^2(x^2 - r_1^2)$. As mentioned above, the derivative of $p(x)$ has the form $p'(x) = dx^{m_0-1}(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - c_1^2)(x^2 - c_2^2)$. Therefore, $q(x) = d(x^2 - c_1^2)(x^2 - c_2^2)$. Expanding both forms of $q(x)$ and equating coefficients, we obtain the relations above. ■

Application. Although the symmetric five root case has not been solved, the above lemma can help reduce the amount of work of a computer search for examples. At the end of this chapter, we give some of these examples we have found using this lemma.

The following results follow from the relations above. Each result can help reduce the amount of work of a computer search for examples.

THEOREM 5.2.2. *Let $p(x) = x^{m_0}(x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ be a nice symmetric or antisymmetric polynomial of degree d with five roots. Suppose the prime p does not divide the degree d. Then at least one of the nonzero roots is not a multiple of p.*

PROOF. Suppose the prime p divides both r_1 and r_2. By (5.2.1), p^2 divides the sum $c_1^2 + c_2^2$ since p does not divide the degree d. By (5.2.2), p divides c_1 or c_2; however, p^2 divides $c_1^2 + c_2^2$, so p divides both c_1 and c_2. This fact contradicts the assumption that $p(x)$ is in reduced form. Therefore, p does not divide both r_1 and r_2. ■

The following two results apply to nice symmetric or antisymmetric polynomials of prime-power degree.

COROLLARY 5.2.3. *Let $p(x)$ be a nice symmetric or antisymmetric polynomial of degree d with five roots. If the degree d is a power of the prime p, then the greatest common divisor of r_1 and r_2 is also a power of p.*

PROOF. Let p_0 be any prime not equal to p. Since p_0 does not divide d, then by Theorem 5.2.2, p_0 does not divide both r_1 and r_2. ■

THEOREM 5.2.4. *Let $p(x)$ be a nice symmetric or antisymmetric polynomial of degree d with five roots. If the degree $d = p^n$ where p is prime, then GCD(r_1, r_2)$= p^m$ where $m \leq n - 1$.*

PROOF. Suppose p^n divides both r_1 and r_2, so p^{4n} divides $r_1^2 r_2^2$. Let $k < n$ be the largest number such that p^k divides m_0. Therefore, by (5.2.2), p^{4n+k} divides $dc_1^2c_2^2$, so $p^{4n+k-n} = p^{3n+k}$ divides $c_1^2c_2^2$. Since $p(x)$ is in reduced form, we can assume, without loss of generality, that p divides c_1. Because p^{2n} divides the left side of (5.2.1), p^{2n} divides $p^n(c_1^2 + c_2^2)$, the right side of (5.2.1). Hence, p^n divides $c_1^2 + c_2^2$. However, p divides c_1 and p divides $c_1^2 + c_2^2$, so p also divides c_2, which contradicts the assumption that $p(x)$ is in reduced form. By Corollary 5.2.3, we conclude that GCD(r_1, r_2)$= p^m$ where $m \leq n - 1$. ■

The following theorem and its corollary apply to types where d/m_0 is a natural number.

THEOREM 5.2.5. *Let $p(x)$ be a nice symmetric or antisymmetric polynomial of degree d with five roots and suppose d/m_0 is a natural number. If there exists a prime p such that p^2 does not divide both $d-2m_1$ and $d-2m_2$, then p does not divide both c_1 and c_2.*

PROOF. In this case, (5.2.2) becomes $r_1r_2 = nc_1c_2$. Suppose there exists a prime p such that p divides c_1 and c_2 and that p^2 does not divide both $d-2m_1$ and $d-2m_2$. Since p^2 divides the sum $c_1^2 + c_2^2$, p^2 divides the left side of (5.2.1). Furthermore, by (5.2.2), p^2 divides r_1r_2. Since $p(x)$ is in reduced form, we can assume, without loss of generality, that p^2 divides r_1. Since p^2 divides the left side of (5.2.1) and r_1, p^2 divides $(d-2m_2)r_2^2$. Since p^2 does not divide $d-2m_2$, by assumption, p divides r_2^2. Hence, p divides r_2. This fact contradicts the assumption that $p(x)$ is in reduced form. Therefore, p does not divide both c_1 and c_2. ■

COROLLARY 5.2.6. *Let $p(x)$ be a nice symmetric or antisymmetric polynomial of degree d with five roots and suppose d/m_0 is a natural number. If $d-2m_1$ and $d-2m_2$ are products of distinct primes, then c_1 and c_2 are relatively prime.*

PROOF. Since $d-2m_1$ and $d-2m_2$ are products of distinct primes, we can apply Theorem 5.2.5 to any prime p. Since every prime p does not divide both c_1 and c_2, we conclude that c_1 and c_2 are relatively prime. ■

Comments on Table 5.7.1. Table 5.7.1 at the end of the chapter classifies nice symmetric or antisymmetric polynomials with five roots. We also include the results of computer searches that have not revealed any examples. In this case, we indicate the largest radius we have checked by writing $> r_0$ for $r_{\min}$.

5.3. Symmetric Six Root Case

Now we consider the case where $p(x) \in \mathbb{Z}[x]$ is a nice symmetric polynomial with six roots. If we center $p(x)$ at the origin, then $p(x)$ has the form $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}(x^2 - r_3^2)^{m_3}$; and the derivative has the form $p'(x) = dx(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - r_3^2)^{m_3-1}(x^2 - c_1^2)(x^2 - c_2^2)$. The following lemma gives the relations between the roots and critical points of all symmetric polynomials with six roots.

LEMMA 5.3.1. *If $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}(x^2 - r_3^2)^{m_3}$ is a symmetric polynomial with integer coefficients and with six integer roots, then $p(x)$ is nice iff there exist integers c_1 and c_2 such that*

$$2(m_1 r_2^2 r_3^2 + m_2 r_1^2 r_3^2 + m_3 r_1^2 r_2^2) = dc_1^2 c_2^2, \tag{5.3.1}$$

$$2[(m_2 + m_3)r_1^2 + (m_1 + m_3)r_2^2 + (m_1 + m_2)r_3^2] = d(c_1^2 + c_2^2). \tag{5.3.2}$$

PROOF. Differentiating $p(x)$ above, we have

$$p'(x) = 2q(x)x(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - r_3^2)^{m_3-1},$$

where $q(x) = m_1(x^2 - r_2^2)(x^2 - r_3^2) + m_2(x^2 - r_1^2)(x^2 - r_3^2) + m_3(x^2 - r_1^2)(x^2 - r_2^2)$. As mentioned above, $p'(x) = dx(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - r_3^2)^{m_3-1}(x^2 - c_1^2)(x^2 - c_2^2)$. Therefore, $q(x) = (d/2)(x^2 - c_1^2)(x^2 - c_2^2)$. Expanding both forms of $q(x)$ and equating coefficients, we obtain the relations above. ■

Application. Although the symmetric six root case has not been solved, the relations above can help reduce the amount of work of a computer search for examples.

Comments on Table 5.7.2. Table 5.7.2 at the end of the chapter classifies all known examples of nice symmetric polynomials with six roots.

5.4. Symmetric Seven Root Case

Now we consider the case where $p(x) \in \mathbb{Z}[x]$ is a nice symmetric or antisymmetric polynomial with seven roots. If we center $p(x)$ at the origin, then $p(x)$ has the form $p(x) = x^{m_0}(x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}(x^2 - r_3^2)^{m_3}$; and the derivative has the form $p'(x) = dx^{m_0-1}(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - r_3^2)^{m_3-1}(x^2 - c_1^2)(x^2 - c_2^2)(x^2 - c_3^2)$. The following lemma gives the relations between the roots and critical points of all symmetric or antisymmetric polynomials with seven roots.

Lemma 5.4.1. *If $p(x) = x^{m_0}(x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}(x^2 - r_3^2)^{m_3}$ is a symmetric or antisymmetric polynomial with integer coefficients and with seven integer roots, then $p(x)$ is nice iff there exist integers c_1, c_2, and c_3 such that*

$$D_1 r_1^2 + D_2 r_2^2 + D_3 r_3^2 = d(c_1^2 + c_2^2 + c_3^2), \tag{5.4.1}$$

$$D_{12} r_1^2 r_2^2 + D_{13} r_1^2 r_3^2 + D_{23} r_2^2 r_3^2 = d(c_1^2 c_2^2 + c_1^2 c_3^2 + c_2^2 c_3^2), \tag{5.4.2}$$

$$m_0 r_1^2 r_2^2 r_3^2 = d c_1^2 c_2^2 c_3^2, \tag{5.4.3}$$

where $D_i = d - 2m_i$ and $D_{ij} = d - 2m_i - 2m_j$.

Proof. Differentiating $p(x)$ above, we have

$$p'(x) = q(x) x^{m_0-1}(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - r_3^2)^{m_3-1},$$

where $q(x) = 2m_3 x^2(x^2 - r_1^2)(x^2 - r_2^2) + 2m_2 x^2(x^2 - r_1^2)(x^2 - r_3^2) + 2m_1 x^2(x^2 - r_2^2)(x^2 - r_3^2) + m_0(x^2 - r_1^2)(x^2 - r_2^2)(x^2 - r_3^2)$. As mentioned above, $p'(x) = dx^{m_0-1}(x^2 -$

$r_1^2)^{m_1-1}(x^2-r_2^2)^{m_2-1}(x^2-r_3^2)^{m_3-1}(x^2-c_1^2)(x^2-c_2^2)(x^2-c_3^2)$. Therefore, $q(x) = d(x^2 - c_1^2)(x^2 - c_2^2)(x^2 - c_3^2)$. Expanding both forms of $q(x)$ and equating coefficients, we obtain the relations above. ∎

Application. Although the symmetric seven root case has not been solved, the relations above can help reduce the amount of work of a computer search for examples.

Comments on Table 5.7.3. Although no examples of nice symmetric or antisymmetric polynomials with seven roots have been found, Table 5.7.3 at the end of this chapter classifies the computer searches for nice symmetric or antisymmetric polynomials with seven roots performed so far. See the explanation for Table 5.7.1 at the end of Section 5.2 for more information on reading this table.

5.5. Nice Polynomials with N Roots

Suppose $p(x) \in \mathbb{Z}[x]$ is a nice polynomial with N roots. If we shift $p(x)$ so that one of its roots is zero, then $p(x)$ has the form $p(x) = x^{m_0}(x-r_1)^{m_1}\dots(x-r_{N-1})^{m_{N-1}}$; and, by definition, the derivative has the form $p(x) = dx^{m_0-1}(x-r_1)^{m_1-1}\dots(x-r_{N-1})^{m_{N-1}-1}(x-c_1)\dots(x-c_{N-1})$. The relations between the roots and critical points of all polynomials with three roots (Lemma 3.2.1) and all polynomials with four roots (Lemma 4.2.1) suggest the following pattern for the relations between the roots and critical points of all polynomials with N roots:

$$\sum_{1\le i\le N-1} D_i r_i = d \sum_{1\le i\le N-1} c_i,$$
$$\sum_{1\le i<j\le N-1} D_{ij} r_i r_j = d \sum_{1\le i<j\le N-1} c_i c_j,$$
$$\sum_{1\le i<j<k\le N-1} D_{ijk} r_i r_j r_k = d \sum_{1\le i<j<k\le N-1} c_i c_j c_k,$$
$$\vdots$$
$$(d - m_1 - \cdots - m_{N-1}) r_1 \cdots r_{N-1} = d c_1 \cdots c_{N-1}$$

where $D_i = d - m_i$, $D_{ij} = d - m_i - m_j$, $D_{ijk} = d - m_i - m_j - m_k$, and so on.

Remark. We have not yet proven this result. See Problem 17.

5.6. Examples

The following nice symmetric or antisymmetric polynomials were found with the help of the results stated above.

Degree 8. These are the first known examples of nice polynomials with six roots:

$$p(x) = (x^2 - 2^2)^2(x^2 - 6^2)(x^2 - 16^2), \tag{5.6.1}$$

$$p'(x) = 8x(x^2 - 2^2)(x^2 - 5^2)(x^2 - 14^2);$$

$$p(x) = (x^2 - 25^2)(x^2 - 7^2)(x^2 - 17^2)^2, \tag{5.6.2}$$

$$p'(x) = 8x(x^2 - 23^2)(x^2 - 17^2)(x^2 - 11^2).$$

The only known example of a nice symmetric octic with five roots is (2.2.5).

Degree 9. The only known example of a nice antisymmetric nonic with five roots is (2.2.4).

Degree 10. The following two examples are inequivalent examples of the same type:

$$p(x) = (x^2-1)^3(x^2-29^2)(x^2-79^2), \tag{5.6.3}$$

$$p'(x) = 10x(x^2-1)^2(x^2-25^2)(x^2-71^2);$$

$$p(x) = (x^2-229^2)^3(x^2-59^2)(x^2-19^2), \tag{5.6.4}$$

$$p'(x) = 10x(x^2-229^2)^2(x^2-43^2)(x^2-149^2).$$

Degree 12. The following example has six roots:

$$p(x) = (x^2-383^2)(x^2-281^2)^3(x^2-167^2)^2, \tag{5.6.5}$$

$$p'(x) = 12x(x^2-281^2)^2(x^2-167^2)(x^2-217^2)(x^2-365^2).$$

Degree 16. The following example has five roots:

$$p(x) = x^4(x^2-38^2)(x^2-682^2)^5, \tag{5.6.6}$$

$$p'(x) = 16x^3(x^2-682^2)^4(x^2-31^2)(x^2-418^2).$$

Degree 18. The following example also has five roots:

$$p(x) = x^2(x^2-15^2)^5(x^2-48^2)^3, \tag{5.6.7}$$

$$p'(x) = 18x(x^2-15^2)^4(x^2-48^2)^2(x^2-6^2)(x^2-40^2).$$

Degree 25. The following example is of the type (8,8,4,4,1).

$$p(x) = x(x^2-175^2)^4(x^2-760^2)^8, \tag{5.6.8}$$

$$p'(x) = 25(x^2-175^2)^3(x^2-760^2)^7(x^2-475^2)(x^2-56^2).$$

The examples given in (3.2.5) and (3.2.6) are inequivalent examples of the type (11,11,1,1,1).

5.7. Open Problems and Conjectures

CONJECTURE 3. Nice symmetric or antisymmetric polynomials of type $(m_0, m_1, m_1, m_1, m_1)$ (if they exist) have larger radii for their sets of roots than nice symmetric or antisymmetric polynomials of type $(m_0, m_1', m_1', m_2', m_2')$ (if they exist) where $m_1' \neq m_2'$ and $m_0 + 4m_1 = m_0 + 2m_1' + 2m_2' = d$.

CONJECTURE 4. Nice antisymmetric nonics of the type (2,2,2,2,1) do not exist. (A computer search up to radius 18,498 has not revealed any examples of this type.)

CONJECTURE 5. Nice symmetric or antisymmetric polynomials of the types (23,23,1,1,1), (6,6,6,6,1), (4,3,3,3,3), (9,4,4,4,4) do not exist. (Computer searches up to radius 4000 to 10,000 or beyond have not revealed any examples of these types.)

CONJECTURE 6. There exists $N > 3$ such that formulas for all nice polynomials with N or more roots cannot be found. In other words, it is impossible to find a formula for all nice polynomials with any specified number of roots.

PROBLEM 8. Prove or disprove Conjecture 3.

PROBLEM 9. Prove or disprove Conjecture 4.

PROBLEM 10. Prove or disprove Conjecture 5.

PROBLEM 11. Prove or disprove Conjecture 6.

PROBLEM 12. Find an example of a nice symmetric or antisymmetric polynomial with five roots where $m_1 = m_2$.

PROBLEM 13. Find a nice symmetric sextic with six distinct roots.

PROBLEM 14. Find a formula for all nice symmetric and antisymmetric polynomials with five roots.

PROBLEM 15. Find a formula for all nice symmetric polynomials with six roots.

PROBLEM 16. Find a formula for all nice symmetric and antisymmetric polynomials with seven roots.

PROBLEM 17. Prove the relations given for all polynomials with N roots. If these given relations are not correct, then find the correct relations.

Classification of Nontrivial Nice Symmetric and Antisymmetric Polynomials with Five Roots					
N_{classes} = Number of equivalence classes N_{examples} = Number of known examples r_{min} = Radius of the smallest example S = Solved U = Unsolved or Unknown					
Degree	Type	Status	N_{classes}	N_{examples}	r_{min}
8	(2,2,2,1,1)	U	U	1 in (2.2.5)	598
9	(3,3,1,1,1)	U	U	1 in (2.2.4)	513
9	(2,2,2,2,1)	U	U	0	$>$ 18498
16	(5,5,4,1,1)	U	U	1 in (5.6.6)	682
16	(4,3,3,3,3)	U	U	0	$>$ 3964
18	(5,5,3,3,2)	U	U	1 in (5.6.7)	48
25	(6,6,6,6,1)	U	U	0	$>$ 3950
25	(11,11,1,1,1)	U	U	2	385 in (3.2.5)
25	(8,8,4,4,1)	U	U	1 in (5.6.8)	760
25	(9,7,7,1,1)	U	U	0	$>$ 2350
25	(9,4,4,4,4)	U	U	0	$>$ 5410
49	(23,23,1,1,1)	U	U	0	$>$ 10192

TABLE 5.7.1. Classification Table for Five Roots

Classification of Nontrivial Nice Symmetric Polynomials with Six Roots					
N_{classes} = Number of equivalence classes N_{examples} = Number of known examples r_{min} = Radius of the smallest example S = Solved U = Unsolved or Unknown					
Degree	Type	Status	N_{classes}	N_{examples}	r_{min}
8	(2,2,1,1,1,1)	U	U	2	16 in (5.6.1)
10	(3,3,1,1,1,1)	U	U	2	79 in (5.6.3)
12	(3,3,2,2,1,1)	U	U	1 in (5.6.5)	383

TABLE 5.7.2. Classification Table for Six Roots

Classification of Nontrivial Nice Symmetric and Antisymmetric Polynomials with Seven Roots					
N_{classes} = Number of equivalence classes N_{examples} = Number of known examples r_{min} = Radius of the smallest example S = Solved U = Unsolved or Unknown					
Degree	Type	Status	N_{classes}	N_{examples}	r_{min}
8	(2,1,1,1,1,1,1)	U	U	0	> 424
18	(8,3,3,1,1,1,1)	U	U	0	> 204
25	(10,10,1,1,1,1,1)	U	U	0	> 495
25	(8,8,2,2,2,2,1)	U	U	0	> 495
49	(10,10,7,7,7,7,1)	U	U	0	> 637
49	(11,11,11,11,2,2,1)	U	U	0	> 644

TABLE 5.7.3. Computer Searches So Far

Classification of Nontrivial Nice Symmetric Sextics				
N_{classes} = Number of equivalence classes N_{examples} = Number of known examples r_{min} = Radius of the smallest example S = Solved U = Unsolved or Unknown				
Type	Status	N_{classes}	N_{examples}	r_{min}
(3,3)	S	1	1	1
(2,2,2)	S	0 by Thm. 2.2.2		
(4,1,1)	S	0 by Thm. 2.2.2		
(2,2,1,1)	S by Thm. 4.3.2	∞	∞	5
(2,1,1,1,1)	S	0 by Thm. 2.2.2		
(1,1,1,1,1,1)	U	U	0	> 856

TABLE 5.7.4. Classification Table for Sextics

Classification of Nontrivial Nice Symmetric Octics				
N_{classes} = Number of equivalence classes N_{examples} = Number of known examples r_{min} = Radius of the smallest example S = Solved U = Unsolved or Unknown				
Type	Status	N_{classes}	N_{examples}	r_{min}
(4,4)	S	1	1	1
(6,1,1)	S	0 by Thm. 3.2.3		
(4,2,2)	S	0 by Thm. 3.2.3		
(3,3,2)	S by Thm. 3.2.3	1	1	2
(2,2,2,2)	S by Thm. 4.3.2	∞	∞	7
(3,3,1,1)	S by Thm. 4.3.2	∞	∞	8
(4,1,1,1,1)	S	0 by Thm. 2.2.2		
(2,2,1,1,1,1)	U	U	2	16
(2,1,1,1,1,1,1)	U	U	0	> 424
(1,1,1,1,1,1,1,1)	U	U	0	

TABLE 5.7.5. Classification Table for Octics

Bibliography

[1] Tom Bruggeman and Tom Gush, Nice cubic polynomials for curve sketching, *Math Magazine* 53(4):233-234, 1980.

[2] Ralph H. Buchholz and James A. MacDougall, When Newton met Diophantes: A study of rational-derived polynomials and their extensions to quadratic fields, *J. Number Theory* 81:210-233, 2000.

[3] Chris K. Caldwell, Nice polynomials of degree 4, *Math. Spectrum* 23(2):36-39, 1990.

[4] M. Chapple, A cubic equation with rational roots such that it and its derived equation also has rational roots, *Bull. Math. Teachers Secondary Schools*, 11: 5-7, 1960, (Recently republished in *Aust. Senior Math. J.* 4(1): 57-60, 1990).

[5] Jean-Claude Evard, Polynomials whose roots and critical points are integers, Submitted and posted on the Website of Arxiv Organization at the address http://arxiv.org/abs/math.NT/0407256.

[6] Richard Nowakowski, Unsolved problems, 1969-1999, *Amer. Math. Monthly* 106(10):959-962, 1999.

[7] Karl Zuser, Uber eine gewisse Klasse von ganzen rationalen Funktionen 3. Grades (in German), *Elem. Math*, 18:101-104, 1963.